

Antimagic labelings of regular bipartite graphs: An application of the Marriage Theorem

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Abstract

A labeling of a graph is a bijection from $E(G)$ to the set $\{1, 2, \dots, |E(G)|\}$. A labeling is *antimagic* if for any distinct vertices u and v , the sum of the labels on edges incident to u is different from the sum of the labels on edges incident to v . We say a graph is antimagic if it has an antimagic labeling. In 1990, Ringel conjectured that every connected graph other than K_2 is antimagic. In this paper, we show that every regular bipartite graph (with degree at least 2) is antimagic. Our technique relies heavily on the Marriage Theorem.

1 Introduction

In this paper, we study a problem of edge-labeling. For convenience, we formally define a *labeling* of a graph G to be a bijection from $E(G)$ to the set $\{1, \dots, |E(G)|\}$. A *vertex-sum* for a labeling is the sum of the labels on edges incident to a vertex v ; we also call this the *sum at v* . A labeling is *antimagic* if the vertex-sums are pairwise distinct. A graph is *antimagic* if it has an antimagic labeling.

Hartsfield and Ringel [2] introduced antimagic labelings in 1990 and conjectured that every connected graph other than K_2 is antimagic. The most significant progress on this problem is a result of Alon, Kaplan, Lev, Roditty, and Yuster [1], which states the existence of a constant c such that if G is an n -vertex graph with $\delta(G) \geq c \log n$, then G is antimagic. Large degrees satisfy a natural intuition: the more edges are present, the more flexibility there is to arrange the labels and possibly obtain an antimagic labeling.

Alon et al. also proved that G is antimagic when $\Delta(G) \geq |V(G)| - 2$, and they proved that all complete multipartite graphs (other than K_2) are antimagic. Hartsfield and Ringel proved that paths, cycles, wheels, and complete graphs are antimagic.

In this chapter, we show that every regular bipartite graph (with degree at least 2) is antimagic. Our proof relies heavily on the Marriage Theorem, which states that every regular bipartite graph has a 1-factor. By induction on the vertex degree, it follows that a regular bipartite graph decomposes into 1-factors. Recall that a k -factor is a k -regular spanning subgraph, so the union of any k 1-factors is a k -factor. Throughout

this paper, we refer to the partite sets of the given bipartite graph as A and B , each having size n .

With respect to a given labeling, two vertices *conflict* if they have the same sum. We view the process of constructing an antimagic labeling as resolving the “potential conflict” for every pair of vertices. We will label the edges in phases. When we have labeled a subset of the edges, we call the resulting sum at each vertex a *partial sum*.

Our general approach is to label all but a single 1-factor so that the partial sums in A are multiples of 3, while the partial sums in B are non-multiples of 3. At this stage no vertex of A conflicts with a vertex of B . We then label the final 1-factor with reserved labels that are multiples of 3 so that we resolve all potential conflicts within A and within B . Before we begin the general approach, we observe two facts that together show that 2-regular graphs are antimagic.

Fact 1. [2] *Every cycle is antimagic.*

Proof: Assign the labels to edges as $1, 3, \dots, n, n-1, \dots, 4, 2$ in order around an n -cycle (if n is odd; otherwise, n and $n-1$ are switched in the middle). The sums are $4, 8, \dots, 10, 6, 3$; that is, the sums of consecutive odd integers are even multiples of 2, while the sums of consecutive even integers are odd multiples of 2. \square

Fact 2. *If G_1 and G_2 are each regular antimagic graphs, then the disjoint union of G_1 and G_2 is also antimagic.*

Proof: Index G_1 and G_2 so that vertices in G_2 have degree at least as large as those in G_1 . Let $m_1 = |E(G_1)|$. Place an antimagic labeling on G_1 , using the first m_1 labels. Label G_2 by adding m_1 to each label in an antimagic labeling of G_2 . Let k be the degree of the vertices in G_2 .

Translating edge labels by m_1 adds $m_1 k$ to the sum at each vertex of G_2 , so the new labeling of G_2 has distinct vertex sums. Hence there are no conflicts within G_1 and no conflicts within G_2 . There are also no conflicts between a vertex in G_1 and one in G_2 , since each vertex-sum in G_1 is less than $m_1 k$ and each vertex-sum in G_2 is greater than $m_1 k$. \square

More generally, given any labeling of a regular graph, adding the same amount to each label does not change the pairs of vertices that conflict. Fact 1 and Fact 2 immediately yield:

Corollary 3. *Every simple 2-regular graph is antimagic.*

For degrees larger than 2, we will consider odd and even degree separately. Although 2-regular graphs are easy, the general construction is a bit more complicated for even degree than for odd degree.

2 Regular bipartite graphs with odd degree

We have observed that a k -regular bipartite graph G decomposes into 1-factors. We can combine these 1-factors in any desired fashion. In particular, when k is odd and at least 5, we can decompose G into a $(2l + 2)$ -factor and a 3-factor, where $l \geq 0$. Our aim will be to combine special labelings of these two factors to obtain an antimagic labeling of G . The case $k = 3$ is handled separately; we do this before the general argument.

Theorem 4. *Every 3-regular bipartite graph is antimagic.*

Proof: Since G has $3n$ edges, we have the same number of labels in each congruence class modulo 3. For convenience, we use the term j -labels to designate the first n positive integers that are congruent to j modulo 3, where $j \in \{0, 1, 2\}$.

Decompose G into a 1-factor H_1 and a 2-factor H_2 . We will reserve the 0-labels for H_1 . We will label H_2 with the 1-labels and 2-labels so that the partial sum at each vertex of A is $3n$. We do this by pairing each 1-label i with the 2-label $3n - i$. These pairs have sum $3n$; at each vertex of A , we use the two integers in some pair. Subsequently, every assignment of 0-labels to H_1 yields distinct vertex-sums within A .

We have assigned a pair of labels at each vertex of A in H_2 , but we have not decided which edge gets which label. Next we try to make this choice so that in H_2 the partial sums at vertices of B will not be multiples of 3. In each component of H_2 , we will fail at most once.

Let \mathcal{C} be a cycle that is a component of H_2 . We have a 1-label and a 2-label at each vertex of A . As we follow \mathcal{C} , if we have a 1-label and then a 2-label at a vertex of A , then the next vertex of A should have a 2-label followed by a 1-label (and vice versa), since the sum of two 1-labels or two 2-labels is not a multiple of 3. If $|V(\mathcal{C}) \cap A|$ is even, then we succeed throughout; if $|V(\mathcal{C}) \cap A|$ is odd, then at one vertex of \mathcal{C} in B we will have a 1-label and a 2-label. Call such a vertex of B *bad*. A cycle in H_2 has a bad vertex only if it has length at least 6, so at most $n/3$ vertices in B will be bad. Let m be the number of bad vertices.

To avoid conflicts between vertices of A and bad vertices of B , we will make the vertex-sum at each bad vertex smaller than at any vertex of A . Furthermore, we will make the partial sums in H_2 at these vertices equal. Consider the 1-labels and 2-labels from 1 through $3m - 1$; group them into pairs j and $3m - j$. The sum in each such pair is $3m$, which is at most n . Allocate the pairs for H_2 to vertices of A so that at each bad vertex of B , the labels are the small elements from pairs in the original pairing and form a pair with sum $3m$ in this most recent pairing.

Now we need to label H_1 . We must achieve three goals: resolve all conflicts among the good vertices in B , resolve all conflicts among the bad vertices in B , and resolve all conflicts between A and the bad vertices in B .

We consider the last goal first. For every assignment of 0-labels to H_1 , the vertex-sums in A will be $\{3n + 3, 3n + 6, \dots, 6n - 3, 6n\}$. To ensure that the vertex-sums at the bad vertices in B will be less than $3n + 3$, we use the smallest 0-labels at the bad vertices. Since there are at most $n/3$ bad vertices, every 0-label at such a vertex is at most n . Thus, every sum at a bad vertex is at most $2n$, which is less than $3n$.

Furthermore, the sums at bad vertices are $3m$ plus distinct 0-labels; hence they are distinct, which completes the second goal.

For the first goal, let b_1, b_2, b_3, \dots denote the good vertices of B in order of increasing partial sum from H_2 (there may be ties). We assign the remaining 0-labels to edges of H_1 at b_1, b_2, \dots in increasing order. Since the 0-labels are distinct, this prevents conflicts among the good vertices in B . \square

For larger even degree, we will construct an antimagic labeling from special labelings of two subgraphs. Like the labeling we constructed for 3-regular graphs, the first labeling will have equal sums at vertices of A , but this time we guarantee that all sums at vertices of B are not congruent modulo 3 to the sums at vertices of A .

Lemma 5. *If G is a $(2l + 2)$ -regular bipartite graph with parts A and B of size n , then G has a labeling such that the sum at each vertex of A is some fixed value t and the sum at each vertex of B is not congruent to t modulo 3.*

Proof: As remarked earlier, we can decompose G into a $2l$ -factor H_{2l} and a 2-factor H_2 . Let $m = (2l + 2)n$; thus m is the largest label. Since m is even, we can partition the labels 1 through m into pairs that sum to $m + 1$. With $m + 1 \equiv 2a \pmod{3}$, each pair consists of two elements in the same congruence class as a modulo 3 or elements in the two other congruence classes modulo 3. Call these *like-pairs* and *split-pairs*, respectively.

At each vertex of A , we will use l of these pairs as labels in H_{2l} . Thus each vertex of A will have partial sum $(m + 1)l$ in H_{2l} ; we will assign the pairs so that the partial sums in B are not congruent to $(m + 1)l$ modulo 3. We use the pairs in which the smaller label ranges from 1 to ln . Note that H_{2l} decomposes into even cycles (for example, we can take $2l$ 1-factors two at a time to generate 2-factors whose union is H_{2l}).

For each cycle in the decomposition of H_{2l} into even cycles, at vertices of A we use pairs of labels of the same type: all like-pairs or all split-pairs. When using split-pairs, we assign the labels so that the same congruence class modulo 3 is always first. If we have all like-pairs or all split-pairs, this ensures that at each vertex of B , each cycle contributes an amount to the sum that is congruent to $2a$ modulo 3. There is at most one cycle where we are forced to use both like-pairs and split-pairs. Let x and y be the vertices of B where, in this cycle, we switch between like-pairs and split-pairs. At each vertex of A , the partial sum in H_{2l} is $(m + 1)l$. At each vertex of B , except x and y , the partial sum is congruent to $(m + 1)l$ modulo 3.

On H_2 , we use the remaining pairs of labels so that we add $m + 1$ to each partial sum in A , but what we add to each partial sum in B is not congruent to $m + 1$ modulo 3. If we can do this (and treat x and y specially), then the sum at each vertex of A will be $(m + 1)(l + 1)$, while at each vertex of B the sum will be in a different congruence class modulo 3 from $(m + 1)(l + 1)$.

On each cycle, we use the pairs of labels that contain the smallest unused labels. Thus, every third pair we use is a like-pair; the others are split pairs. We begin with a like-pair and alternate using a like-pair and a split-pair until the like-pairs allotted to that cycle are exhausted. For the remaining split-pairs, we alternate them in the form $(a + 1, a + 2)$ followed by $(a + 2, a + 1)$; in this way the sum of the two labels used at

any vertex of B is not congruent to $2a$ modulo 3. If no like-pair is available to be used on the cycle, then the cycle has length 4 and we label it with split-pairs in the form $(a + 1, a + 2), (a + 2, a + 1)$, and the same property holds.

One or two cycles in H_2 may contain the vertices x and y , where the sum in H_{2l} differs by 1 from a value congruent to $(m + 1)l$ modulo 3. Suppose that the sums in H_{2l} at x and y are $(m + 1)l + t_1$ and $(m + 1)l + t_2$. We want the sum at x in H_2 to be either $2a - t_1 + 1 \pmod{3}$ or $2a - t_1 + 2$. Similarly, we want the sum at y in H_2 to be in $\{2a - t_2 + 1, 2a - t_2 + 2\}$. The more difficult case is when x and y lie on the same cycle in H_2 . However, given the realization that we have two choices each for the sums (modulo 3) at x and y , it is not difficult to adapt the labeling given above for cycles of H_2 so that it applies in the current case as well.

At these vertices we want the contribution from H_2 to be congruent to $2a$ modulo 3. We deal with these first and can then make the argument above for the remaining cycles. If x and y lie on a single 4-cycle, then we use two like-pairs or two split-pairs ordered as $(a + 1, a + 2), (a + 1, a + 2)$. If one or both of x and y lie on a longer cycle, then at each we put edges from two like-pairs or from two split-pairs ordered as $(a + 1, a + 2), (a + 1, a + 2)$. The remaining pairs, whether they are like-pairs or split-pairs as we allocate them to this cycle, can be filled in so that like-pairs are not consecutive anywhere else and neighboring split-pairs alternate their “orientation”.

Thus the labeling of H_2 enables us to keep the overall sum at each vertex of B out of the congruence class of $(m + 1)(l + 1)$ modulo 3. \square

Lemma 6. *If G is a 3-regular bipartite graph with parts A and B , where $B = \{b_1, \dots, b_n\}$, then G has a labeling so that at each b_i the sum is $3n + 3i$, and for each i exactly one vertex in A has sum $3n + 3i$.*

Proof: Decompose G into three 1-factors: R , S , and T . In R , use label $3i - 2$ on the edge incident to b_i ; let a_i be the other endpoint of this edge. In S , use label $3n + 3 - 3i$ on the edge incident to a_i ; call the other endpoint of this edge b'_i . In T , use label $3i - 1$ on the edge incident to b'_i ; call the other endpoint of this edge a'_i . Note that each 1-factor received the labels from one congruence class modulo 3.

The partial sum in $S \cup T$ at each vertex of B is $3n + 2$. Hence, the sum at b_i for all of G is $3n + 3i$. Similarly, the partial sum in $R \cup S$ at each vertex of A is $3n + 1$. Hence, the vertex-sum at a'_i is $3n + 3i$. \square

Theorem 7. *Every regular bipartite graph of odd degree is antimagic.*

Proof: Let G be a regular bipartite graph of degree k . Theorem 4 is the case $k = 3$. For $k > 3$, let $k = 2l + 5$ with $l \geq 0$, and decompose the graph G into a 3-factor H_3 and a $(2l + 2)$ -factor H_{2l+2} . Label H_{2l+2} as in Lemma 5; this uses labels 1 through $(2l + 2)n$. Add $3n$ to each label, leaving labels 1 through $3n$ for H_3 . Each vertex-sum increases by $9n$, which is a multiple of 3, so the congruence properties obtained in Lemma 5 remain true for the new labeling.

Let b_i denote the vertices of B in order of increasing partial sum in H_{2l+2} . Label H_3 as in Lemma 6. Because all the partial sums in H are multiples of 3, the labeling of H_{2l+2} resolves each potential conflict between a vertex of A and a vertex of B . Because

the b_i are in order of increasing partial sum in H_{2l+2} , the labeling of H_3 resolves all potential conflicts within B . Similarly, since the labeling of H_{2l+2} gives the same partial sum to all vertices of A , the labeling of H_3 resolves all potential conflicts within A .

We have checked that the labeling is antimagic. \square

3 Regular bipartite graphs with even degree

Lemma 8. *Let n be a positive integer. If n is even, then we can partition $\{1, 2, \dots, 3n\}$ into triples such that the sum of each triple is $6n + 3$ or $3n$. If n is odd, then we can partition $\{1, 2, \dots, 3n\}$ into triples such that the sum of each triple is $6n$ or $3n$. Furthermore, each triple consists of one integer from each residue class modulo 3.*

Proof: Suppose n is even. We partition the labels into triples so that the sum of each triple is either $3n$ or $6n + 3$. Consider the triples $(3n - 3i + 3, 3n - 3i + 2, 6i - 2)$ and $(3i, 3i - 1, 3n - 6i + 1)$ for $1 \leq i \leq n/2$. Triples of the first type sum to $6n + 3$ and triples of the second type sum to $3n$.

Suppose n is odd. We partition the labels into triples so that the sum of each triple is either $3n$ or $6n$. Consider the triples $(3n - 3i + 3, 3n - 3i + 2, 6i - 5)$ for $1 \leq i \leq \lceil n/2 \rceil$ and $(3i, 3i - 1, 3n - 6i + 1)$ for $1 \leq i \leq \lfloor n/2 \rfloor$. Triples of the first type sum to $6n$ and triples of the second type sum to $3n$. \square

Theorem 9. *Every regular bipartite graph of even degree at least 8 is antimagic.*

Proof: We decompose G into two 3-factors and a $(2l + 2)$ -factor; call these G_3, H_3 , and H_{2l+2} , respectively. We label H_{2l+2} as in Lemma 5, using all but the $6n$ smallest labels. This resolves every conflict between a vertex of A and a vertex of B .

We partition the labels $\{3n + 1, 3n + 2, \dots, 6n\}$ into triples as in Lemma 8. In G_3 , at each vertex of A we will use the three labels of some triple. To ensure the sum at each vertex of B is $0 \pmod{3}$, we do the following. Partition the 3-factor into three 1-factors; We use 0-labels on the first 1-factor, 1-labels on the second 1-factor, and 2-labels on the third 1-factor.

Now consider the partial sums in the union of H_{2l+2} and G_3 ; let b_i denote the vertices of B in order of increasing partial sum. Label H_3 as in Lemma 6. This resolves every conflict between two vertices in the same part. Hence, the labeling is antimagic. \square

Lemma 10 is very similar to Lemma 8. Lemma 10 serves the same role in the proof of Theorem 11 that Lemma 8 does in the proof of Theorem 9.

Lemma 10. *Let n be a positive integer. Let H be the set of positive labels less than $4n$ that are not 0 modulo 4, i.e. $H = \{1, 2, 3, 5, 6, \dots, 4n - 2, 4n - 1\}$. If n is even, then we can partition H into triples such that the sum of each triple is either $4n - 2$ or $8n + 2$. If n is odd, then we can partition H into triples such that the sum of each triple is either $4n - 2$ or $8n - 2$. Furthermore, each triple consists of one integer from each nonzero residue class modulo 4.*

Proof: Suppose n is even. We have triples of the form $(8i - 3, 4n - 4i + 2, 4n - 4i + 3)$, with $1 \leq i \leq n/2$, and triples of the form $(4n - 8i + 1, 4i - 2, 4i - 1)$, with $1 \leq i \leq n/2$. It is easy to see that triples of the first form sum to $8n + 2$ and that triples of the second form sum to $4n - 2$. It is straightforward to verify that these triples partition H .

Suppose n is odd. We have triples of the form $(8i - 7, 4n - 4i + 2, 4n - 4i + 3)$, with $1 \leq i \leq \lceil n/2 \rceil$, and triples of the form $(4n - 8i + 1, 4i - 2, 4i - 1)$, with $1 \leq i \leq \lfloor n/2 \rfloor$. It is easy to see that triples of the first form sum to $8n - 2$ and that triples of the second form sum to $4n - 2$. It is straightforward to verify that these triples partition H . \square

Theorem 11. *Every 6-regular bipartite graph is antimagic.*

Proof: Throughout this proof we assume that n is odd. The argument is analogous when n is even, so we omit the details. We decompose G into a 1-factor, a 2-factor, and a 3-factor; call these H_1 , H_2 , and H_3 , respectively. We label H_3 with the labels that are less than $4n$ and are not 0 modulo 4, so that the partial sum at each vertex of B is $2 \pmod{4}$ and the partial sum at each vertex of A is $4n - 2$ or $8n - 2$. To do this we partition the labels for H_3 into triples as specified in Lemma 10.

At each vertex of A , we use the three labels in some triple. More exactly, we decompose H_3 into three 1-factors; we use $1 \pmod{4}$ labels on the first 1-factor, use $2 \pmod{4}$ labels on the second 1-factor, and use $3 \pmod{4}$ labels on the third 1-factor. This ensures that the partial sum at each vertex of B is $2 \pmod{4}$.

We label H_2 with the labels $4n + 1$ through $6n$, so that the partial sum at each vertex of A is $10n + 1$ and the sum at each vertex of B is $\not\equiv 10n + 1 \pmod{4}$. To do this, we partition the labels for H_2 into pairs that sum to $10n + 1$. We consider the labels in each pair modulo 4. We have two types of pairs: $(1, 2)$ pairs and $(3, 0)$ pairs (since n is odd).

We want to avoid using two labels at a vertex of B that sum to $3 \pmod{4}$. We choose the pairs of labels to use on each cycle arbitrarily, except that each cycle must use at least one $(1, 2)$ pair and at least one $(3, 0)$. We first use all the $(1, 2)$ pairs, alternating them as $(1, 2), (2, 1), (1, 2), (2, 1), \dots$, then use all the $(3, 0)$ pairs, alternating them as $(3, 0), (0, 3), (3, 0), (0, 3), \dots$. As long as we use at least one $(1, 2)$ pair and one $(3, 0)$ pair on each cycle of H_2 , we have no problems. Since we use at least one $(1, 2)$ pair and one $(3, 0)$ pair on each cycle of H_2 , we are able to avoid vertex sums in B that are congruent to $3 \pmod{4}$.

Now we consider partial sums in $H_2 \cup H_3$. The partial sum at each vertex of A is $14n - 1$ or $18n - 1$. The partial sum at each vertex of B is not congruent to 3 modulo 4. The labels we will use on the H_1 are all multiples of 4. Hence, regardless of how we label H_1 , no vertex in A will conflict with any vertex in B . We call a vertex in A *small* if its partial sum in $H_2 \cup H_3$ is $14n - 1$; otherwise, we call it *big*. It is clear that regardless of how we label H_1 , no big vertex will conflict with another big vertex; similarly, no small vertex will conflict with a small vertex. Observe that the largest possible sum at a small vertex is $14n - 1 + 4n = 18n - 1$. The smallest possible sum at a big vertex is $18n - 1 + 4 = 18n + 3$. Hence, no small vertex will conflict with a big vertex. Thus, we choose the labels for H_1 to ensure that no two vertices in B conflict.

Let b_i denote the vertices of B in order of increasing partial sum in $H_2 \cup H_3$. In H_1 , we use label $4i$ at vertex i . This ensures that vertex-sums in B are distinct. Thus, the labeling is antimagic. \square

The proof for 4-regular graphs is more complicated than for 6-regular graphs. In the 6-regular case, we labeled the 2-factor to ensure there were no conflicts between any vertex in A and any vertex in B ; we labeled the 1-factor and the 3-factor to ensure there were no conflicts between two vertices in the same part. The proof for 4-regular graphs is similar, but since we have one less 2-factor, we cannot ensure that all vertex-sums in B differ modulo 4 from the vertex-sums in A . So similar to the 3-regular graphs, we introduce *good* and *bad* vertices in B . We handle bad vertices in a similar way to the case of the 3-regular graphs.

Theorem 12. *Every 4-regular bipartite graph is antimagic.*

Proof: Throughout this proof we assume that n is odd. The argument is analogous when n is even, so we omit the details. We decompose G into a 1-factor H_1 and a 3-factor H_3 . We label H_3 with the labels that are less than $4n$ and are not 0 modulo 4, so that the partial sum at each vertex of B is $4n - 2$ or $8n - 2$. To do this, we partition the labels for the 3-factor into triples as specified in Lemma 10. At each vertex in A , we will use the three labels of a triple. Consider a vertex of B : if its partial sum in the 2-factor is $2 \pmod{4}$, then we call the vertex *bad*; otherwise, we call it *good*. We assign the labels of each triple to the edges at a vertex of A to minimize the number of bad vertices in B . Initially, we only assign to each edge a residue class: $1 \pmod{4}$, $2 \pmod{4}$, or $3 \pmod{4}$. This determines which vertices in B are bad. We will then assign the labels to edges to minimize the largest partial sum at a bad vertex of B . Since the bad vertices in B will have vertex-sums in the same residue class (modulo 4) as the vertex-sums in A , to avoid conflicts we will ensure that the vertex-sum at every bad vertex is smaller than the smallest vertex-sum in B .

We begin by decomposing H_3 into three 1-factors. We label each edge in the first 1-factor with a 1, each edge in the second 1-factor with a 2, and each edge in the third 1-factor with a 3. However, this makes every vertex in B bad. To fix this, we consider the 2-factor labeled with 1s and 2s; specifically consider a single cycle in this 2-factor. Select a vertex of A on the cycle, then select every second vertex of A along the cycle; at each of the selected vertices, swap the labels 1 and 2 on the incident edges. If the cycle has length divisible by 4, then all of its vertices are now good. If the length is not divisible by 4, then one bad vertex will remain. Note that a cycle has a bad vertex only if its length is at least 6. So, at most $n/3$ vertices are bad. We now reduce the number of bad vertices further, as follows.

If a vertex is bad, consider the incident edge labeled 3, and the edge labeled 2 that is adjacent in A to this first edge; these two edges form a *bad path*. We will swap the two labels on a bad path to reduce the number of bad vertices. Consider the graph induced by bad paths; each component is a path or a cycle. In a path component, we swap the labels on every second bad path; this fixes all the bad vertices. We handle cycle components similarly, although in each cycle one bad vertex may remain (similar to the previous step). Thus, after this step, at most $1/3$ of the previously bad vertices

remain bad. So, at most $n/9$ vertices remain bad. We also need to verify that when we swap the labels on a bad path, no good vertex becomes bad.

If a good vertex has partial sum $3 \pmod{4}$, we call it *heavy*; if it has partial sum $1 \pmod{4}$, we call it *light*. Before we swap the labels on any bad path, the triple of labels incident to a light vertex is $(1, 1, 3)$; the triple incident to a heavy vertex is $(2, 2, 3)$. Thus, we do not swap any labels incident to a light vertex. However, the labels incident to a heavy vertex could become $(2, 3, 3)$ or even $(3, 3, 3)$. In each case though, the vertex remains good.

Finally, if any vertex in A is adjacent to two or more bad vertices, we swap the labels on its incident edges to make each vertex good. Thus, we have at most $n/9$ bad vertices and each vertex in A is adjacent to at most one bad vertex. Now we assign the actual labels to the edges (rather than only the residue classes) so that the partial sum at each bad vertex is small. We assign the $n/9$ smallest $1 \pmod{4}$ labels to be incident to the bad vertices; the largest is less than $4n/9$. Similarly, we assign the $n/9$ smallest $2 \pmod{4}$ labels to be incident to the bad vertices; again the largest is less than $4n/9$. Each time we assign a label, we also assign the other labels in its triple. Since each $2 \pmod{4}$ label is in a triple with the $3 \pmod{4}$ label one greater, the $n/9$ smallest $3 \pmod{4}$ labels are already assigned. So we assign the next $n/9$ smallest $3 \pmod{4}$ labels to be incident to the bad vertices; the largest of these labels is less than $8n/9$. Finally, we will assign the $n/9$ smallest $0 \pmod{4}$ labels to be incident to the bad vertices. Thus, the largest vertex-sum at a bad vertex is less than $3(4n/9) + 8n/9 < 3n$. Hence, no bad vertex will conflict with any vertex in A .

To ensure that no two bad vertices conflict, we assign the labels to the final 1-factor in order of increasing partial sum at the bad vertices. After we assign all the labels incident to the bad vertices, we assign the remaining labels incident to the good vertices, again in order of increasing partial sum in B . This ensures that no two good vertices conflict. If the partial sum at a vertex of A is $4n - 2$ we call it *small*; otherwise we call it *big*. After we assign the labels on the final 1-factor, the smallest possible vertex-sum at a big vertex is $(8n - 2) + 4 = 8n + 2$; the largest possible sum at a small vertex is $(4n - 2) + 4n = 8n - 2$. So no small vertex conflicts with a big vertex. Additionally, all the small vertex-sums are distinct; so are the large vertex-sums. Thus, the labeling is antimagic. \square

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